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The renormalisation group and global G × G′ theories about four dimensions

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Abstract. The critical behaviour of linear Φ^4 models with global symmetry $O(N) \times O(M)$ and $U(N) \times U(M)$ is studied at one-loop order in $4-\varepsilon$ dimensions. Applications to physical systems include achiral and chiral double-strand polymers, when $N = M = 0$. There exist infrared stable fixed points only if N and M are sufficiently small. Some special cases of interest include (1) $O(-2) \times O(1)$ and $U(-1) \times U(1)$ which have classical exponents, (2) $O(1) \times O(1)$ and $O(-2) \times O(-2)$ (for example) which exhibit a 'merging' of critical exponents, (3) M finite, $N \to \infty$ which is also calculable in a $1/N$ expansion.

1. Introduction

Vector Φ^4 models with global $O(N)$ symmetry have been the subject of much study (Wilson and Fisher 1972, Brézin *et al* 1973, 1974). Natural generalisations include higher representations of a simple group (Amit 1976, Priest and Lubensky 1976, Amit and Roginsky 1979) and semisimple groups. Previously we sketched the results of an expansion in $4-\varepsilon$ dimensions for the semisimple groups $O(N) \times O(M)$ and $U(N) \times U(M)$ in the vector representation (Pisarski and Stein 1981). In this work we consider these models in detail.

The results are obtained from an expansion to lowest order in ε. In broad outline they are analogous to those found for *NM* models (Brézin *et al* 1974, Aharony 1976). There are always four fixed points, of which one and only one is infrared (IR) stable for sufficiently small values of N and M. For many values of N and M two of the fixed points become complex, and there is no IR stable fixed point. There is always an IR stable fixed point in the *NM* model, although its absence is common to numerous related systems (Aharony 1976, Bak *et al* 1976).

There are several physical systems related to our model. $N = M = 0$ corresponds to double-strand achiral and chiral polymers for $O(N) \times O(M)$ and $U(N) \times U(M)$ (de Gennes 1972). Similar to an $O(-2)$ vector (Balian and Toulouse 1973), we find that the critical exponents for $O(-2) \times O(1)$ and $U(-1) \times U(1)$ are classical. $U(1) \times U(1)$ models are used to describe a $^3P_2$ superfluid in a strong magnetic field (Hoffberg *et al* 1970). At $M = 0$, $N \neq 0$ our model must correspond to some sort of quenched random magnet, in analogy to the same limit in the *NM* model (Aharony 1976, Grinstein and Luther 1976).

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The paper is organised thus. In § 2 we discuss $N=M=0$ and multi-strand polymers. Section 3 reviews the renormalisation group techniques needed (Brézin *et al* 1974). We show in § 4 how an N^{-1} expansion about $N=\infty$, M finite and non-zero, allows a simple check. The critical behaviour for $O(N)\times O(M)$ and $U(N)\times U(M)$ is considered in §§ 5 and 6. We discuss in particular small values of $N=M$. For example, occasionally critical values exhibit identities between η and γ, even with four distinct fixed points. This merging of critical values (not points) is easy to understand for $O(1)\times O(1)$ and $U(1)\times U(1)$ but less obvious for $O(-2)\times O(-2)$ and $U(-1)\times U(-1)$. Lastly, the critical limit for $N\to\infty$, both for $M\neq 0$ and $M=0$, is given in § 7. The instance $N\to\infty$ and $M=0$ is of note as there are two IR stable fixed points, only one of which is physical.

Paterson (1980) has also studied $U(N)\times U(N)$ models below four dimensions, with results that overlap ours.

2. $N=M=0$ and double-strand polymers

de Gennes (1972) first noticed that the space–time path of an $O(N)$ vector field can be directly identified with a single-strand polymer if $N=0$. Φ^4 interactions represent the repulsive self-interaction of a long polymer bending back on itself. The caveat $N=0$ is necessary to eliminate virtual Φ loops which occur for quantum fields.

The propagator for an $O(N)\times O(M)$ vector $\boldsymbol{\Phi}$ is drawn as two dissimilar lines. The global symmetry restricts the renormalised Lagrangian density (equation (3.1)) to be a sum of bilinears in $\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}$. The two couplings are proportional to $(\mathrm{Tr}\,\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi})^2$, with strength g_1, and $\mathrm{Tr}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi})^2$, with strength g_2. Two typical vertices for g_1 and g_2 are illustrated in figures 1(a) and 1(b) with many other terms generated by the full Dyson–Wick expansion. Note that the coupling of figure 1(b) is peculiar to the symmetry being semisimple, in that if $g_2=0$, the coupling of figure 1(a) is that for an $O(N^*M)$ vector.

(a) (b)

Figure 1. (a) $(\mathrm{Tr}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}))^2$ vertex. (b) $\mathrm{Tr}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi})^2$ vertex.

Hence an $O(N)\times O(M)$ vector at $N=M=0$ corresponds to a polymer ribbon, where the ribbon is one of two dissimilar strands. For example, figure 1(a) illustrates two ribbons which touch without the exchange of strands. Figure 1(b) shows two ribbons which touch and entangle by exchanging strands between the ribbons.

Calculation (§§ 5, 6 and 7) shows that at the critical point the ribbons completely untangle: the infrared stable fixed point is that for an $O(N)$ vector at $N=0$, $g_2^*=0$. We emphasise that the identity between polymers of a single-strand and multi-strand ribbons holds only at the critical point, where the polymer is infinitely long. For any physical polymer of finite length, non-universal terms will show the entanglement of ribbons.

That very long ribbons untangle themselves is easily understood. It is simple for a ribbon to hit itself as in figure 1(*a*). In contrast, the probability of figure 1(*b*) occurring vanishes as the ribbon becomes infinitely long, since figure 1(*b*) represents the exchange of two very long strands between ribbons.

Shortly we give many examples of multi-strand ribbons, related to the $N = 0$ limits for tensor fields of $O(N)$ and $U(N)$, etc. In all cases we expect long ribbons to untangle, with the fixed point of an O(0) vector. Again, the properties of finite ribbons will depend on the number, similarity and chirality of the strands in the ribbon.

Ribbons with similar strands are formed from the tensors of $O(N)$. Tensors of second rank are given by $N \times N$ matrices $\mathbf{\Phi}$ which transform with definite sign under transposition of indices, either (1) $\mathbf{\Phi}$ symmetric, $\mathrm{Tr}\,\mathbf{\Phi} = p$ ($p = -N, \ldots, +N$), or (2) $\mathbf{\Phi}$ antisymmetric. For the polymer, transposition is just the exchange of strands in the ribbon. Hence a symmetric field at $N = 0$ describes a double-strand ribbon with two similar strands. An antisymmetric field is unphysical at $N = 0$, as the exchange of strands gives -1. Generally, tensor fields interact through terms as $\mathrm{Tr}\,\mathbf{\Phi}^3$. However, for a polymer $\mathrm{Tr}\,\mathbf{\Phi}^3$ is unphysical, as it represents strands which appear or disappear.

Proceeding to the unitary group allows strands to have chirality, such as results from the helical structure of a strand. Thus a $(U(N))^k$ vector at $N = 0$ represents a k-strand ribbon, where all strands are dissimilar but have the same chirality. The tensors of $U(N)$ are chosen from matrices which have definite sign under transposition and complex conjugation. The three second-rank tensors are given by (1) $\mathbf{\Phi}$ Hermitian, $\mathrm{Tr}\,\mathbf{\Phi} = p$, (2) $\mathbf{\Phi}$ symmetric and (3) $\mathbf{\Phi}$ antisymmetric. For a polymer, complex conjugation reverses the chirality of each strand. At $N = 0$ the double-strand ribbons are then (1) two strands of opposite chirality, (2) two strands of the same chirality (as for the double helical structure of DNA) and (3) unphysical.

Lastly, there is the $N = 0$ limit of symplectic fields. Whatever chirality exists for symplectic tensors at $N = 0$, it must reflect that the series of classical Lie groups $O(N)$, $U(N)$, $Sp(N)$ ends with the symplectic group.

3. Renormalisation group machinery

For an $O(N) \times O(M)$ vector field $\mathbf{\Phi}$ the bare Lagrangian is taken as

$$L = \tfrac{1}{2}\,\mathrm{Tr}(\partial_\mu \mathbf{\Phi}^{\mathrm{T}})(\partial_\mu \mathbf{\Phi}) + \frac{8\pi^2\mu^\varepsilon}{4!}[g_1(\mathrm{Tr}\,\mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi})^2 + g_2\,\mathrm{Tr}(\mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi})^2] + (\mathrm{Tr}\,\mathbf{\Phi}^{\mathrm{T}}\mathbf{\Phi})H(x). \qquad (3.1)$$

The bare field (Φ) and couplings ($g_{1,2}$) are related to their renormalised values (denoted by a tilde) as

$$\Phi = Z_3\tilde{\Phi}, \qquad g_{1,2} = \tilde{g}_{1,2}/Z_3^2. \qquad (3.2)$$

Expressions for the critical values $\beta_{1,2}$, η and γ are given by Brézin *et al* (1973).

In general, there are four fixed points where $\beta_i^* = 0$:

$$\begin{aligned} g_\alpha^* &= (0, 0), \\ g_\beta^* &= (6\varepsilon/(NM+8), 0) \qquad \text{for } O(N)\times O(M), \\ g_\beta^* &= (3\varepsilon/(NM+4), 0) \qquad \text{for } U(N)\times U(M), \end{aligned} \qquad (3.3)$$

and two others, $g_{\delta^+}^*$ and $g_{\delta^-}^*$. This fixed point g_β^* become complex for N and M sufficiently large (§§ 5, 6, 7).

To illustrate the conditions for the stability of a fixed point, assume for the moment only one coupling constant. A running coupling $\tilde{g}(\lambda)$ is given by

$$\beta(\tilde{g}) = \mathrm{d}\,\tilde{g}(\lambda)/\mathrm{d}\ln\lambda.$$

If we define the subcritical exponent ω as $\omega = \mathrm{d}\beta/\mathrm{d}\tilde{g}\,|\,\tilde{g} = \tilde{g}^*$, then for $\tilde{g}$ near $\tilde{g}^*$

$$\tilde{g}(\lambda) - \tilde{g}^* = \lambda^{\omega}. \tag{3.4}$$

In the infrared limit, $\lambda \to 0$, so that for an infrared stable (IRS) fixed point $\omega > 0$. With L coupling constants, ω becomes an $L \times L$ matrix such that a fixed point is IRS when the eigenvalues w of ω satisfy Re $w > 0$.

Note that all one-loop results, and ω in general, are independent of renormalisation convention.

We present in §§ 5, 6, 7 a detailed analysis for the domains of attraction to fixed points for $\mathrm{O}(N) \times \mathrm{O}(M)$ and $\mathrm{U}(N) \times \mathrm{U}(M)$ in the ultraviolet (UV) and IR limits. We invariably found it easiest to determine the domains of attraction by numerical analysis.

The fixed point structure in the UV is simple: g_α^* is always stable when $\varepsilon > 0$. This is just the statement that $\boldsymbol{\Phi}^4$ theories are asymptotically free in $4-\varepsilon$ dimensions. Let the domain of attraction in the UV to g_α^* be defined as D_{UV}. In D_{UV}, if a field Φ is scaled by $\Phi \to \kappa\Phi$, then the interaction term in the renormalised Lagrangian, L_{int}, will behave as

$$L_{\mathrm{int}} \sim \kappa^{4-\varepsilon}[g_1(\kappa)(\mathrm{Tr}\,\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi})^2 + g_2(\kappa)\,\mathrm{Tr}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi})^2]. \tag{3.5}$$

The quartic form L_{int} is positive in the region R^+, where $R^+ = \{g_2 > 0, g_1 + g_2/N > 0\}$. In the region D^+, which is the intersection of D_{UV} and R^+, equation (3.5) for L_{int} establishes the boundedness of the energy from below. It is to be emphasised that the condition $g_{1,2} \in D^+$ is a sufficient but *not* a necessary condition for the energy to be bounded from below.

The distinction is significant in understanding the IR fixed point structure. If there exists an IRS fixed point, as is true when N and M are not too large, there is always in addition a twice unstable g_α^* and two saddle points. The IRS g^* can be either g_β^*, $g_{\delta^+}^*$ or $g_{\delta^-}^*$, but is 'usually' g_β^* (§§ 5, 6, 7). Now *if* there exists an IRS fixed point, a domain of attraction in the IR limit, D_{IR}, can be defined. For example, if we set the renormalised coupling g_2 equal to zero by fiat, the $\mathrm{O}(M) \times \mathrm{O}(N)$ symmetry becomes $\mathrm{O}(M \times N)$. Since there is an IRS fixed point for all N in an $\mathrm{O}(N)$ model (in a vector representation), D_{IR} will *always* include the g_1 axis. Hence at least in the intersection of D^+ and D_{IR} ($D^* \equiv D^+ \cap D_{\mathrm{IR}}$, where D^* is in general some section of the first quadrant), we can confidently claim perturbation theory is valid in both the UV and IR limits.

In many instances there is no IRS fixed point. In the IR limit, the critical trajectories are driven out of D^+ into the second quadrant. As we are driven into a region where the energy is not bounded from below, the stable ground state will not be $\boldsymbol{\Phi} = 0$. Thus due to quantum fluctuations the phase transition will be first order (Bak *et al* 1976).

The first-order phase transition in $\mathrm{U}(N) \times \mathrm{U}(N)$, for $N > 2$, has also been studied by Paterson (1980).

4. An N^{-1} expansion of $\mathrm{O}(N) \times \mathrm{O}(M)$

In this section we work out the critical limit of $\mathrm{O}(N) \times \mathrm{O}(M)$ for $N \to \infty$ with M fixed by an N^{-1} expansion, in a manner familiar from the vector model (Brézin and Zinn-Justin 1978).

To begin with, we shall find it useful to redefine the coupling constants and explicitly introduce a mass term so that the bare Lagrangian is given by

$$L=\tfrac{1}{2}\,\mathrm{Tr}[(\partial_\mu \boldsymbol{\Phi}^{\mathrm{T}})(\partial_\mu \boldsymbol{\Phi})+m^2\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi}]+\tfrac{1}{2}G_1^2(\mathrm{Tr}\,\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi})^2+\tfrac{1}{2}G_1^2\,\mathrm{Tr}(\boldsymbol{\Phi}^{\mathrm{T}}\boldsymbol{\Phi})^2. \tag{4.1}$$

Auxiliary fields ω_1 and $\boldsymbol{\omega}_2$, where $\boldsymbol{\omega}_2$ is an $M\times M$ matrix, are introduced to integrate out the original $\boldsymbol{\Phi}$ fields and obtain

$$L_{\mathrm{eff}}=\tfrac{1}{2}N\,\mathrm{Tr}\ln[(-\partial^2+m^2)\delta^{ij}+2G_1\omega_1\delta^{ij}+2G_2\omega_2^{ij}]-\tfrac{1}{2}\omega_1^2-\tfrac{1}{2}(\omega_2^{ij})^2. \tag{4.2}$$

We then expand about a stationary point with constant field $\bar\omega_1$ and $\bar\omega_2\delta^{ij}$,

$$\omega_1=\bar\omega_1+\omega_1^{qu},\qquad \omega_2^{ij}=\bar\omega_2\delta^{ij}+\omega_2^{qu,ij}. \tag{4.3}$$

The renormalised mass $\bar m$, which is a finite quantity, is given by

$$\bar m^2=m^2+2G_1\bar\omega_1+2G_2\bar\omega_2. \tag{4.4}$$

Coupling constant renormalisation will also be required. Note that the UV regularisation of the theory requires the physical mass to be non-zero.

At a stationary point of $\bar\omega_1$ and $\bar\omega_2$,

$$\frac{\delta L_{\mathrm{eff}}}{\delta\bar\omega_1}=0\colon\ 2NM\,\mathrm{Tr}\frac{1}{-\partial^2+\bar m^2}=\frac{2\bar\omega_1}{G_1}, \tag{4.5a}$$

$$\frac{\delta L_{\mathrm{eff}}}{\delta\bar\omega_2}=0\colon\ 2N\,\mathrm{Tr}\frac{1}{-\partial^2+\bar m^2}=\frac{2\bar\omega_2}{G_2}. \tag{4.5b}$$

Mass renormalisation is imposed by the conditions

$$m^2/(G_1)^2=\bar m^2/(G_1^*)^2-2NM\,\mathrm{Tr}(1/-\partial^2) \tag{4.6a}$$

and

$$m^2/(G_2)^2=\bar m^2/(G_2^*)^2-2N\,\mathrm{Tr}(1/-\partial^2). \tag{4.6b}$$

Using the identity

$$\mathrm{Tr}\frac{1}{-\partial^2+\bar m^2}-\mathrm{Tr}\frac{1}{-\partial^2}=-\bar m^2\,\mathrm{Tr}\frac{1}{(-\partial^2+\bar m^2)(-\partial^2)},$$

the stationary point equations (equations (4.5)) become

$$\frac{1}{(G_1)^2}=\frac{1}{(G_1^*)^2}-2NM\,\mathrm{Tr}\frac{1}{(-\partial^2+\bar m^2)(-\partial^2)}+\frac{2G_2\bar\omega_2}{(G_1)^2\bar m^2} \tag{4.7a}$$

and

$$\frac{1}{(G_2)^2}=\frac{1}{(G_2^*)^2}-2N\,\mathrm{Tr}\frac{1}{(-\partial^2+\bar m^2)(-\partial^2)}+\frac{2G_1\bar\omega_1}{(G_2)^2\bar m^2}. \tag{4.8a}$$

The critical point is determined by those values for G_1^* and G_2^* such that

$$1/G_1=1/G_2=0.$$

These two coupled equations give the fixed points in the limit $N\to\infty$, $M\neq 0$:

$$g_\beta^*=(6\varepsilon/NM,0),\qquad g_{\delta^+}^*=(0,6\varepsilon/N),\qquad g_{\delta^-}^*=(-6\varepsilon/NM,6\varepsilon/N). \tag{4.9}$$

The unitary group is treated similarly.

5. O(N)×O(M) critical behaviour

Using the Lagrangian of equation (3.1), we find to one-loop order

$$\begin{aligned}\tilde{g}_1 &= g_1+[\tfrac{1}{6}(NM+8)g_1^2+\tfrac{1}{3}(N+M+1)g_1g_2+\tfrac{1}{2}g_2^2]a,\\ \tilde{g}_2 &= g_2+[2g_1g_2+\tfrac{1}{6}(N+M+4)g_2^2]a,\end{aligned} \tag{5.1}$$

where $a=1/\varepsilon$. Hence

$$\begin{aligned}\beta_1 &= \varepsilon g_1+[\tfrac{1}{6}(NM+8)g_1^2+\tfrac{1}{3}(N+M+1)g_1g_2+\tfrac{1}{2}g_2^2],\\ \beta_2 &= -\varepsilon g_2+[2g_1g_2+\tfrac{1}{6}(N+M+4)g_2^2].\end{aligned} \tag{5.2}$$

The first contribution to wavefunction renormalisation occurs at two-loop order

$$Z_3=1+\left(\frac{N+M+2}{18}g_1^2+\frac{N+M+1}{9}g_1g_2+\frac{NM+N+M+3}{36}g_2^2\right)b \tag{5.3}$$

so that

$$\gamma_3=\frac{NM+2}{72}g_1^2+\frac{N+M+1}{36}g_1g_2+\frac{NM+N+M+3}{144}g_2^2, \tag{5.4}$$

with $b=-1/8\varepsilon$ and $\eta=\gamma_3^*$. Finally, we calculate the renormalisation constant associated with insertions of the operator Tr $\Phi^{\mathrm{T}}\Phi$:

$$Z_4=1+[\tfrac{1}{6}(NM+2)g_1+\tfrac{1}{6}(N+M+1)g_2]a, \tag{5.5}$$

so that

$$\gamma=1+\tfrac{1}{12}(NM+2)g_1^*+\tfrac{1}{12}(N+M+1)g_2^*. \tag{5.6}$$

Finally, we calculate the stability matrix ω_{ij} to find

$$\omega=\begin{pmatrix}\dfrac{NM+8}{3}g_1^*+\dfrac{N+M+1}{3}g_2^*-\varepsilon & \dfrac{N+M+1}{3}g_1^*+g_2^* \\ 2g_2^* & 2g_1^*+\dfrac{N+M+4}{3}g_2^*-\varepsilon\end{pmatrix} \tag{5.7}$$

where the real part of both eigenvalues for ω must be positive for the fixed point to be IRS.

Besides g_α^*, which is always UV stable, g_β^* is IR stable for $NM<4$. The two other fixed points are given by the formula

$$(g_1^*)_{\delta^\pm}=\tfrac{1}{2}[\varepsilon-\tfrac{1}{6}(N+M+4)(g_2^*)_{\delta^\pm}], \tag{5.8}$$

$$\begin{aligned}(g_2^*)_{\delta^\pm}/\varepsilon &= 6[(NM-10)(N+M+4)+36]\pm 36[(N+M)^2-12NM-4(N+M)+52]^{1/2}\\ &\quad\times[(NM-16)(N+M)^2+8(NM-7)(N+M+2)+576]^{-1}.\end{aligned}$$

One immediately notices that the $g_{\delta^\pm}^*$ will be complex when

$$(N+M)^2+52<12NM+4(N+M). \tag{5.9}$$

If both this equation and $NM>4$ are satisfied, then there is no IRS fixed point. Note that $g_{\delta^\pm}^*$ will be real if the limit $N\to\infty$, M finite is taken (§ 7).

We illustrate the critical behaviour of this O(M)×O(N) model by considering the special cases when $M=N=-3,\ldots,+3$ (see table 1). It is interesting to see that the

Table 1. $O(N) \times O(N)$ critical exponents for $N = -3, \ldots, +3$. C denotes the quantity as complex.

N	$\frac{g_\beta^*}{\varepsilon}$	$\frac{g_{\delta^+}^*}{\varepsilon}$	$\frac{g_{\delta^-}^*}{\varepsilon}$	$\frac{\eta_\beta}{\varepsilon^2}$	$\frac{\eta_{\delta^+}}{\varepsilon^2}$	$\frac{\eta_{\delta^-}}{\varepsilon^2}$	$\left(\frac{\gamma_\beta - 1}{\varepsilon}\right)$	$\left(\frac{\gamma_{\delta^+} - 1}{\varepsilon}\right)$	$\left(\frac{\gamma_{\delta^-} - 1}{\varepsilon}\right)$
3	$(\frac{6}{17}, 0)$	C	C	$\frac{11}{578}$	C	C	$\frac{11}{34}$	C	C
2	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, 0)$	$(\frac{9}{10}, -\frac{3}{5})$	$\frac{1}{48}$	$\frac{1}{48}$	$\frac{1}{50}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{20}$
1	$(\frac{2}{3}, 0)$	$(\frac{1}{3}, \frac{1}{3})$	$(1, -1)$	$\frac{1}{54}$	$\frac{1}{54}$	$\sim O(\varepsilon)$	$\frac{1}{6}$	$\frac{1}{6}$	$\sim O(\varepsilon)$
0	$(\frac{3}{4}, 0)$	$(0.331, 0.508)$	$(0.704, -0.611)$	$\frac{1}{64}$	0.013	0.0094	$\frac{1}{8}$	0.0975	0.0664
−1	$(\frac{2}{3}, 0)$	$(0.381, 0.713)$	$(0.549, -0.294)$	0.0185	0.005	0.0183	$\frac{1}{6}$	0.036	0.162
−2	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, 0)$	$(\frac{1}{2}, 1)$	$\frac{1}{48}$	$\frac{1}{48}$	$\sim O(\varepsilon)$	$\frac{1}{4}$	$\frac{1}{4}$	$\sim O(\varepsilon)$
−3	$(\frac{6}{17}, 0)$	$(\frac{3}{5}, \frac{3}{5})$	$(\frac{9}{13}, \frac{15}{13})$	$\frac{11}{578}$	$\frac{1}{50}$	$\frac{3}{169}$	$\frac{11}{34}$	$\frac{3}{10}$	$\frac{2}{13}$

critical behaviour is strongly dependent on N within these values. Indeed, the behaviour of the theory for all other values of M and N is well illustrated by the cases below. One atypical example, which is considered in some detail in § 7, is that of $N \to \infty$, M finite.

There are two questions which will be considered in the analysis. One is simply where the renormalisation group trajectories flow in the IR limit; hence, is D^* of finite measure in the (g_1, g_2) plane? The other is to test various conjectures about inequalities between critical exponents. Brézin *et al* (1974) have conjectured that the greatest value of η corresponds to that of the IR *stable* fixed point. We find this violated in the limit $N \to \infty$, $M < 1$, but not for many other values of N and M; in particular, the conjecture is easily verified for all cases in table 1. Analogously, it is reasonable to ask whether the greatest value of $(\gamma - 1)/\varepsilon$ always corresponds to the IRS fixed point. This is disproved in the circumstances $N = M = -3$.

We now comment on the examples of table 1.

$N = -3$: The case is interesting because it is the only integer (albeit negative) for which the IRS fixed point is not g^*_β, but is rather $g^*_{\delta^-}$. Consequently, the structure of D^* is distinctive for this class of models (figure 2). The structure of D^* for $N = -3$ is characteristic of all N in that it is a curvilinear region of the first quadrant with g^*_α, g^*_β and $g^*_{\delta^\pm}$ as the four 'corners' of D^*. The flow of critical trajectories in the infrared limit can be similarly understood. The four fixed points form an approximate rectangle. In

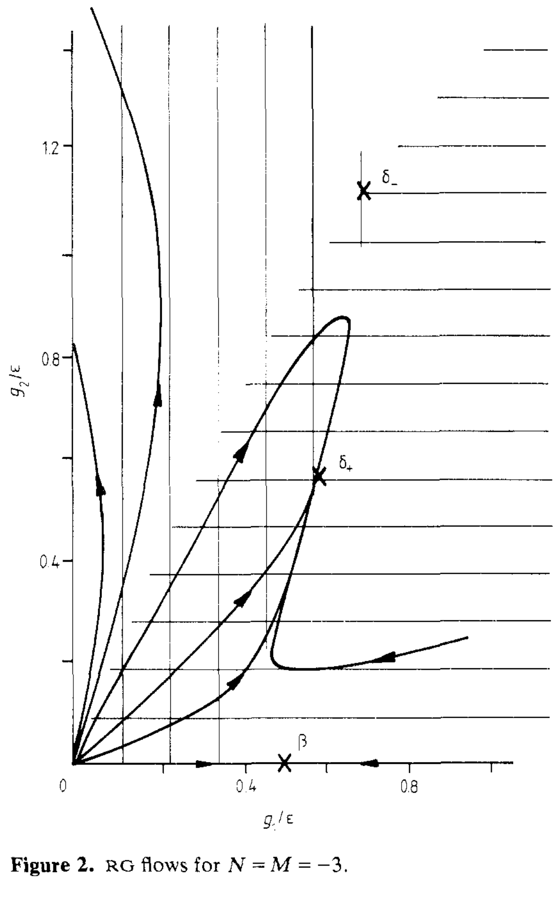

Figure 2. RG flows for $N = M = -3$.

this rectangle, the absolute maxima (g_α^*) and minima (the IRS fixed point) lie on a diagonal, flanked at the remaining corners by the two saddle-point fixed points. Some reflection allows one to conclude that *exactly* this structure of D^* *must* be true in order for these to be an IRS fixed point.

$N = -2$: The stability of the fixed point g_β^* can be seen from the ω matrix only when N^2 is *strictly* less than four; hence $N = \pm 2$ are boundary points for the IR stability of g_β^*. At $N = \pm 2$, g_β^* coincides with $g_{\delta^+}^*$. The stability matrix evaluated at g_β^* has one zero eigenvalue, so that in order to determine the IR stability of g_β^*, we must look directly at the $(\tilde{g}_1, \tilde{g}_2)$ trajectories in the IR limit. This is provided by figure 3, in which the stability of g_β^* $(=g_{\delta^+}^*)$ is immediately seen.

There is one other comment to be made about $N = -2$. At this order, there is a 'merging' of critical exponents:

$$g_\beta^* = g_{\delta^+}^*, \qquad \eta_{\delta^-} = \eta_\alpha(=0), \qquad \gamma_{\delta^-} = \gamma_\alpha(=1). \tag{5.10}$$

It would be interesting to see whether $g_{\delta^-}^*$ possesses classical exponents to all orders in ε.

$N = -1$: This is qualitatively similar to $N = 0$.

$N = 0$: The IRS fixed point is g_β^*; D^* was given in Pisarski and Stein (1981). The stability of g_β^* proves the identity of critical indices for the vector representations of $\mathrm{O}(N)$ and $\mathrm{O}(N) \times \mathrm{O}(N)$ at $N = 0$.

$N = +1$: The critical trajectories and D^* are similar to those of $N = 0$. A point of note is that, to this order in ε,

$$\eta_\beta = \eta_{\delta^+}, \qquad \gamma_\beta = \gamma_{\delta^+}, \qquad \eta_{\delta^-} = \eta_\alpha(=0), \qquad \gamma_{\delta^-} = \gamma_\alpha(=1). \tag{5.11}$$

It is straightforward to show that these identities persist to all orders in ε. For $N = M = 1$, and only then, the Lagrangian (3.1) is equivalent to an $\mathrm{O}(1)$ vector field with coupling $g^+ = g_1^* + g_2^*$. All (isotropic) critical indices must be a function only of g^+. The above identities then follow directly from

$$g_\beta^+ = g_{\delta^+}^+, \qquad g_{\delta^-}^+ = g_\alpha^+(=0). \tag{5.12}$$

The same identities can be proved for $\mathrm{U}(1) \times \mathrm{U}(1)$: see also $\mathrm{U}(-1) \times \mathrm{U}(-1)$.

$N = +2$: Similar to $N = -2$, ω_{ij} has one zero eigenvalue, so that the explicit trajectories must be examined. In contrast to $N = -2$, we see from figure 4 that the fixed point g_β^* $(=g_{\delta^+}^*)$ is IR unstable, D_{IR} is then of measure zero and there is no IRS fixed point. (In fact, this difference is not unexpected. In both cases, $g_\beta^* = g_{\delta^+}^*$. When $N = -2$, the IR unstable fixed point $g_{\delta^-}^*$ is in the first quadrant. This ensures that D_{IR} extends into the first quadrant, so that D^* is of finite measure. When $N = +2$, $g_{\delta^-}^*$ is in the fourth quadrant, so D_{IR} does not extend into the first quadrant, and D^* consists only of the positive $\tilde{g}_1$ axis. D_{IR} will of course have a finite measure in the fourth quadrant, but this is outside the region of interest.)

$N = +3$: This case is typical for all $N \gtrsim 2.1$: the $g_{\delta^\pm}^*$ are complex, so there is no IRS fixed point. The coupling constant trajectories are similar to those for $N = +2$: starting near the $\tilde{g}_1$ axis in the first quadrant, the trajectories are driven off into the depths of the second quadrant.

Returning now to the general $\mathrm{O}(M) \times \mathrm{O}(N)$ model, we note one other case of interest; namely the theory with classical (gaussian) exponents. In the vector $\mathrm{O}(N)$ model, this occurs for $N = -2$, which has a non-trivial stable fixed point at $g^* = \varepsilon$ whose exponents are strictly classical to all orders in $\varepsilon : \eta = 0$, $\gamma = 1$ (Balian and Toulouse 1973, Emery 1975). For $\mathrm{O}(N) \times \mathrm{O}(M)$, we see from equations (5.4) and (5.6) that the

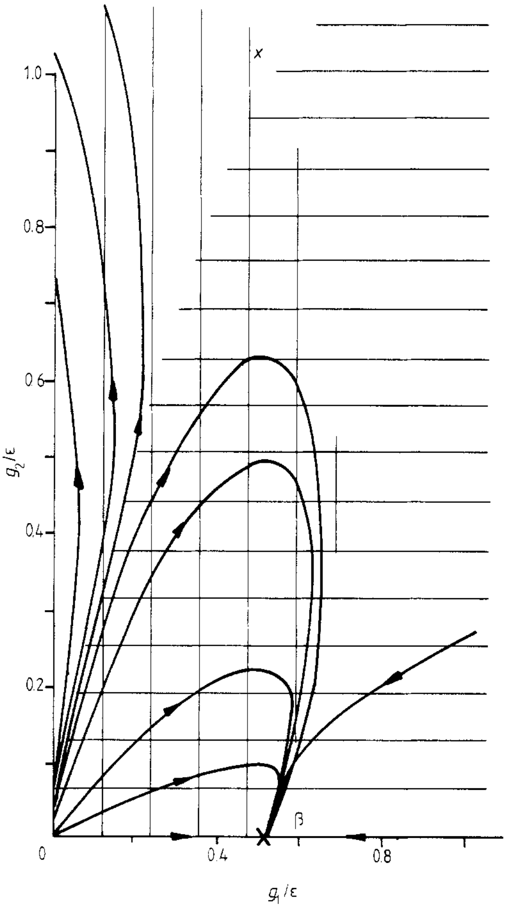

Figure 3. RG flows for $N=M=-2$; note g^*_β is IRS.

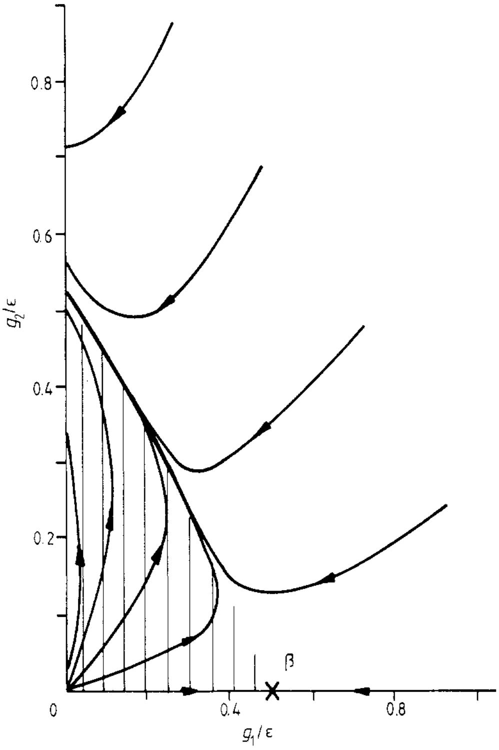

Figure 4. RG flows for $N=M=+2$; note g^*_β is not IRS.

exponents of the stable fixed point are gaussian to lowest order in ε for $NM=-2$, when the stable fixed point is the vector one. Upon closer examination, it becomes clear that *all* fixed points will have gaussian exponents for $N=-2, M=1$ or $N=1, M=-2$. This will in fact be true to all orders in ε. The stable fixed point here is again $g^*_\beta=(\varepsilon, 0)$ and the unstable fixed points are $g^*_{\delta^+}=(\frac{1}{3}\varepsilon, \frac{2}{3}\varepsilon)$ and $g^*_{\delta^-}=(\frac{2}{3}\varepsilon, -\frac{2}{3}\varepsilon)$. It is of interest to note that $(g^*_{\delta^+})_1+(g^*_{\delta^+})_2=(g^*_\beta)_1+(g^*_\beta)_2$ and similarly for g^*_α and $g^*_{\delta^-}$.

6. U(N) × U(M) critical behaviour

In all cases, the critical behaviour of U(N) × U(M) will be very similar to that of O(N) × O(M). Consequently, we shall be brief. The renormalised couplings are found to be

$$\begin{aligned}\tilde{g}_1 &= g_1+[\tfrac{1}{3}(NM+4)g_1^2+\tfrac{2}{3}(N+M)g_1g_2+g_2^2]a,\\ \tilde{g}_2 &= g_2+[2g_1g_2+\tfrac{1}{3}(N+M)g_2^2]a,\end{aligned} \tag{6.1}$$

leading to the β functions

$$\begin{aligned}\beta_1 &= -\varepsilon g_1 + \tfrac{1}{3}(NM+4)g_1^2 + \tfrac{2}{3}(N+M)g_1 g_2 + g_2^2, \\ \beta_2 &= -\varepsilon g_2 + 2g_1 g_2 + \tfrac{1}{3}(N+M)g_2^2.\end{aligned} \tag{6.2}$$

The wavefunction renormalisation is

$$Z_3 = 1 - [\tfrac{1}{72}(NM+1)g_1^2 - \tfrac{1}{36}(N+M)g_1 g_2 - \tfrac{1}{72}(NM+1)g_2^2]b \tag{6.3}$$

from which we find

$$\gamma_3 = \tfrac{1}{36}(NM+1)g_1^2 + \tfrac{1}{18}(N+M)g_1 g_2 + \tfrac{1}{36}(NM+1)g_2^2. \tag{6.4}$$

Finally,

$$Z_4 = 1 + [\tfrac{1}{3}(NM+1)g_1 + \tfrac{1}{3}(N+M)g_2]a, \tag{6.5}$$

so

$$\gamma = 1 + \tfrac{1}{6}(NM+1)g_1^* + \tfrac{1}{6}(N+M)g_2^*. \tag{6.6}$$

The stability matrix ω_{ij} is

$$\omega_{ij} = \begin{pmatrix} \dfrac{2NM+8}{3}g_1^* + \dfrac{2N+2M}{3}g_2^* - \varepsilon & \dfrac{2N+2M}{3}g_1^* + 2g_2^* \\ 2g_2^* & 2g_1^* + \dfrac{2N+2M}{3}g_2^* - \varepsilon \end{pmatrix}. \tag{6.7}$$

The theory will again possess an UV stable fixed point g_2^* at the origin. The 'vector' fixed point is $g^* = (3\varepsilon/(NM+4), 0)$, which is IRS when $NM < 2$. The two other fixed points $g_{\delta^\pm}^*$ are given by the equations

$$\begin{aligned}(g_1^*)_{\delta^\pm} &= \tfrac{1}{2}[\varepsilon - \tfrac{1}{3}(N+M)(g_2^*)_{\delta^\pm}], \\ \frac{(g_2^*)_{\delta^\pm}}{\varepsilon} &= \frac{3(N+M)(NM-5) \pm 9[(N+M)^2 - 12(NM-2)]^{1/2}}{(N+M)^2(NM-8)+108}.\end{aligned} \tag{6.8}$$

The $g_{\delta^\pm}^*$ will be real if

$$(N+M)^2 \geqslant 12(NM-2). \tag{6.9}$$

When this is not satisfied, there is no IRS fixed point within perturbation theory.

We illustrate the critical behaviour for $\mathrm{U}(N)\times\mathrm{U}(M)$ by $N = M = -1, 0, +1, +2$ in table 2. When $N = -1$, 0 and $+1$, g_β^* is the IRS fixed point; $g_{\delta^+}^*$ or $g_{\delta^-}^*$ are never IRS fixed points. When $N = -2$, g_β^* is not IRS, and there is no IRS fixed point. The critical trajectories and D^* for $\mathrm{U}(N)\times\mathrm{U}(N)$ with $N = -1$, 0, 1 and 2 are very similar to those for $\mathrm{O}(N)\times\mathrm{O}(N)$ with these values of N. For the case $\mathrm{U}(1)\times\mathrm{U}(1)$ the identities (equation (5.11)) are true, just as in the $\mathrm{O}(1)\times\mathrm{O}(1)$ case. The IR stability of g_β^* when $N = 0$ demonstrates the equivalence at the critical point between a vector $\mathrm{U}(0)\times\mathrm{U}(0)$ and a vector $\mathrm{O}(0)$ theory.

A merging of critical exponents similar to that for $N = 1$ occurs when $N = -1$. At $N = -1$

$$\eta_\beta = \eta_{\delta^-}, \qquad \gamma_\beta = \gamma_{\delta^-}, \qquad \eta_{\delta^+} = \eta_\alpha(=0), \qquad \gamma_{\delta^+} = \gamma_\alpha(=1). \tag{6.10}$$

These identities can be proved to all orders in ε by the following simple argument. We assume that the only difference between a $\mathrm{U}(-1)$ and a $\mathrm{U}(+1)$ theory is that for

Table 2. U(N)×U(N) critical exponents for $N=-1,\ldots,2$.

N	$\frac{g_\beta^*}{\varepsilon}$	$\frac{g_{\delta^+}}{\varepsilon}$	$\frac{g_{\delta^-}}{\varepsilon}$	$\frac{\eta_\beta}{\varepsilon^2}$	$\frac{\eta_{\delta^+}}{\varepsilon^2}$	$\frac{\eta_{\delta^-}}{\varepsilon^2}$	$\left(\frac{\gamma_\beta-1}{\varepsilon}\right)$	$\left(\frac{\gamma_{\delta^+}-1}{\varepsilon}\right)$	$\left(\frac{\gamma_{\delta^-}-1}{\varepsilon}\right)$
2	$(\frac{3}{8}, 0)$	C	C	$\frac{25}{256}$	C	C	$\frac{5}{16}$	C	C
1	$(\frac{3}{5}, 0)$	$(\frac{9}{20}, \frac{3}{30})$	$(\frac{3}{4}, -\frac{3}{4})$	$\frac{1}{50}$	$\frac{1}{50}$	$\sim O(\varepsilon)$	$\frac{1}{5}$	$\frac{1}{5}$	$\sim O(\varepsilon)$
0	$(\frac{3}{4}, 0)$	$(\frac{1}{2}, 1/\sqrt{6})$	$(\frac{1}{2}, -1/\sqrt{6})$	$\frac{1}{64}$	$\frac{5}{432}$	$\frac{5}{432}$	$\frac{1}{8}$	$\frac{1}{12}$	$\frac{1}{12}$
−1	$(\frac{3}{5}, 0)$	$(\frac{3}{4}, \frac{3}{4})$	$(\frac{9}{20}, -\frac{3}{20})$	$\frac{1}{50}$	$\sim O(\varepsilon)$	$\frac{1}{50}$	$\frac{1}{5}$	$\sim O(\varepsilon)$	$\frac{1}{5}$

$N=-1$, Tr $1=-1$. Consequently, we then write the Lagrangian for the $\mathrm{U}(-1)\times\mathrm{U}(-1)$ theory in terms of a single U(1) field with one coupling constant $g^-=g_1-g_2$. Hence the identities (equations (6.10)) are a consequence of the relations

$$g_\beta^- = g_{\delta^-}^-, \qquad g_\alpha^- = g_{\delta^+}^-.$$

Finally we note that the isotropic critical exponents are free to all orders in ε for $\mathrm{U}(1)\times\mathrm{U}(-1)$, as for $\mathrm{O}(-2)$.

7. Critical behaviour when $N\to\infty$

Up to this point we have discussed $\mathrm{G}\times\mathrm{G}'$ models only for small values of N and M. We now examine the limit for $\mathrm{O}(N)\times\mathrm{O}(M)$ where one index N is taken to infinity with M fixed; the unitary group behaves in much the same way. The fixed points were calculated in this limit (for $M\neq 0$) by means of a $1/N$ expansion in § 4. With $M\neq 0$, $g^*_{\delta^+}$ is the IRS fixed point. One feature of the limit $N\to\infty$ is that D_{IR} includes the entire first quadrant. A representative example of the critical trajectories and D^* is given in figure 5.

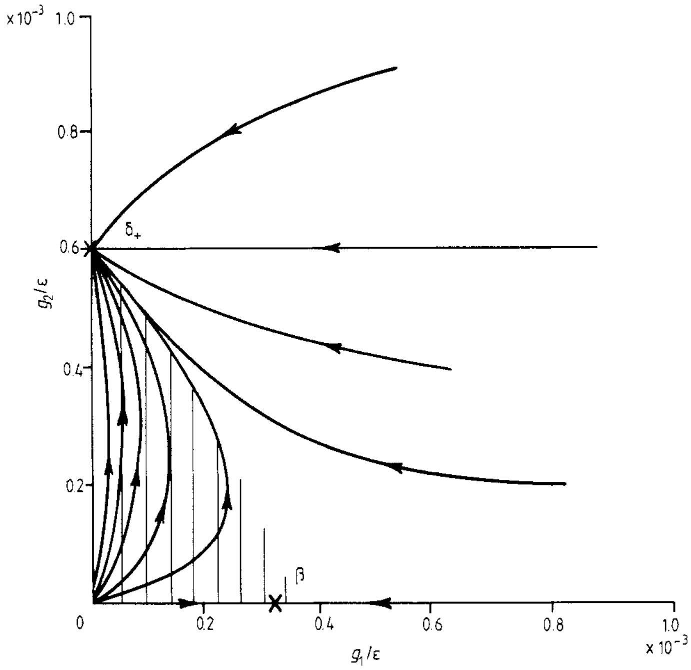

Figure 5. RG flows for $M=2$, $N=10\,000$.

The critical exponents in this limit are given by

$$\eta_\beta=\frac{1}{2NM}\varepsilon^2, \qquad \eta_{\delta^+}=\frac{M+1}{4N}\varepsilon^2, \qquad \eta_{\delta^-}=\left(\frac{M+1}{4N}-\frac{1}{2NM}\right)\varepsilon^2,$$

$$\gamma_\beta=1+\tfrac{1}{2}\varepsilon, \qquad \gamma_{\delta^+}=1+\tfrac{1}{2}\varepsilon, \qquad \gamma_{\delta^-}=1.$$

The greatest η does not belong to the IRS fixed point when $M<1$. Further, to leading order in ε and $1/N$ but for arbitrary (non-zero) M, these critical exponents satisfy the merging relations

$$\eta_\beta+\eta_{\delta^-}=\eta_{\delta^+}+\eta_\alpha, \qquad \gamma_\beta=\gamma_{\delta^+}, \qquad \gamma_{\delta^-}=\gamma_\alpha. \tag{7.1}$$

An identical set of merging relations is found for the unitary group.

The special case $N\to\infty$, $M=0$ is of particular interest since to one-loop order it appears that, contrary to all other examples presented here, there are *two* IRS fixed points. It is easy to appreciate intuitively why the fixed points $g^*_{\delta^+}$ and g^*_β must both be IRS at $N\to\infty$, $M=0$. To begin with, consider beginning at M small but non-zero. The position of $g^*_{\delta^+}$ is unaffected by setting $M=0$. Indeed, critical trajectories in the $(\tilde g_1, \tilde g_2)$ plane near $g^*_{\delta^+}$ for $M\neq 0$ would be expected to behave similarly as $M\to 0$, so $g^*_{\delta^+}$ remains an IRS fixed point as $M\to 0$. On the other hand, we know that g^*_β is IRS for $NM<2$; clearly if M is *strictly* zero then g^*_β is IRS for all N, including $N\to\infty$.

The critical trajectories and D^* are shown for $N\to\infty$, $M=0$ in figure 6. Besides the IRS g^*_β and $g^*_{\delta^+}$, there is the stagnation point $g^*_{\delta^-}$. The fixed point $g^*_{\delta^-}$ is not quite a saddle point: there is one positive eigenvalue for ω_{ij} and one zero eigenvalue at $g^*_{\delta^-}$. The existence of one stagnation point is necessary if there are to be two IRS fixed points.

The existence of two IRS fixed points is *not* necessarily a counterexample to universality. Aharony *et al* (1976) have studied the effects of the cumulants of the disorder distribution in the *NM* model and have concluded that the *NM* counterpart to our vector fixed point (g^*_β) is unphysical in that it can never be reached under RG iterations. The point is that the initial values of $\tilde g_1$ and $\tilde g_2$ must always lie in the domain of IR attraction of $g^*_{\delta^+}$, so that the critical properties of the theory are described by the exponents of $g^*_{\delta^+}$.

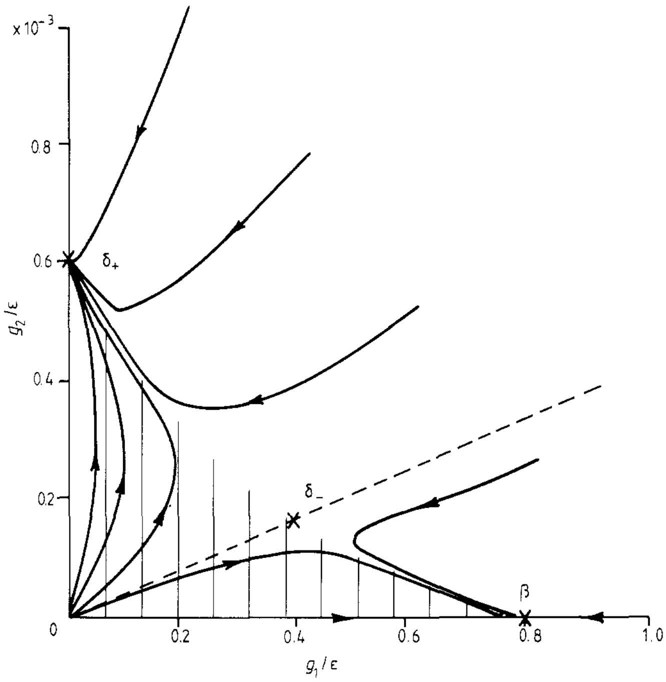

Figure 6. RG flows for $M=0$, $N=10\,000$.

The critical indices for $N \to \infty$, $M = 0$ are

$$\begin{aligned} &\eta_\beta = \tfrac{1}{64}\varepsilon^2, \qquad \eta_{\delta^+} = (1/4N)\varepsilon^2, \qquad \eta_{\delta^-} = \tfrac{1}{256}\varepsilon^2, \\ &\gamma_\beta = 1 + \tfrac{1}{8}\varepsilon, \qquad \gamma_{\delta^+} = 1 + \tfrac{1}{2}\varepsilon, \qquad \gamma_{\delta^-} = 1 + \tfrac{3}{16}\varepsilon. \end{aligned} \tag{7.2}$$

Only the critical exponents of $g^*_{\delta^+}$ appear to be approaching those of the spherical model; g^*_β clearly has a new set of critical exponents.

Finally, we would like to discuss the critical behaviour in the planar limit $N = M \to \infty$. In order that planar graphs dominate as $N \to \infty$, it is natural to scale the coupling constants as

$$\tilde{g}_1 = a/N^2, \qquad \tilde{g}_2 = b/N.$$

In the limit $N \to \infty$ there are three non-trivial fixed points, two complex, and the real fixed point is IR unstable:

$$g^*_\beta : a = 6,\ b = 0, \qquad g^*_{\delta^+} : a = 3(-1 \pm \sqrt{2}\mathrm{i}),\ b = 3;$$

for $\mathrm{O}(N) \times \mathrm{O}(N)$. Hence the planar theory is not calculable perturbatively in the IR limit.

Acknowledgments

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